PHYS 705: Classical Mechanics

General Announcement:

- Final Exam (Scheduled on Dec 13, 2021 Monday 4:30-7:10p)
  - Moving earlier to Dec 06 Monday (Study Day) 4:30-7:10p?

General Notes:

- Not all problems are corrected. Check online solution!
- Play attention to total-time derivative and indices !!!

Total-time Derivative

not just one q

Given: $g\left(q_1,\cdots,q_n,t\right)$ is a function of n generalized variables and time

$$\frac{dg}{dt}\left(q_1,\cdots,q_n,t\right)=\frac{dg}{dt}\left(q_k,t\right),\quad k=1,\cdots,n$$

$$\frac{dg}{dt}\left(q_k,t\right)=\sum_k\frac{\partial g}{\partial q_k}\dot{q}_k+\frac{\partial g}{\partial t}$$

$$=\frac{\partial g}{\partial q_k}\dot{q}_k+\frac{\partial g}{\partial t}$$

n terms in this sum

In Einstein's notations, repeated indices in a product means sum

Indices

$$f_j\left(q_k,t\right)\rightarrow\left\{\begin{matrix} f_1\left(q_1,\cdots,q_n,t\right) \\ f_2\left(q_1,\cdots,q_n,t\right) \\ \vdots \end{matrix}\right\}$$

j -> a given index for a particular j-th f_j (free index)

k -> a running index to indicate f_j depends on ALL the q's (dummy index)

$$\frac{\partial}{\partial q_j}\left(\frac{dg}{dt}\left(q_k,t\right)\right)=\frac{\partial}{\partial q_j}\left(\sum_k\frac{\partial g}{\partial q_k}\dot{q}_k+\frac{\partial g}{\partial t}\right)$$

j -> a given fixed index

k -> a running index

$$=\frac{\partial}{\partial q_j}\left(\frac{\partial g}{\partial q_k}\dot{q}_k+\frac{\partial g}{\partial t}\right)$$

Einstein's notation: repeated dummy indices imply sum

KEEP THEM DISTINCT!

$$F\left(q_j, t\right)$$

More on Indices and Derivatives:

#1: $\dfrac{\partial \dot{F}}{\partial \dot{q}_j}$ → Don't just cancel the dot !!!

$$\frac{\partial \dot{F}}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\left(\frac{dF}{dt}\right) = \frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial F}{\partial t} + \sum_k \frac{\partial F}{\partial q_k}\dot{q}_k\right)$$

$$= \cancel{\frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial F}{\partial t}\right)} + \frac{\partial}{\partial \dot{q}_j}\left(\sum_k \frac{\partial F}{\partial q_k}\dot{q}_k\right)$$

$$= \sum_k \frac{\partial F}{\partial q_k}\frac{\partial \dot{q}_k}{\partial \dot{q}_j} + \sum_k \cancel{\frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial F}{\partial q_k}\right)}\dot{q}_k$$

$$\dot{F} \equiv \frac{dF}{dt} \left(\textit{total time derivative}\right)$$

Since F is not an explicit function of $\dot{q}_j$

$$\frac{\partial \dot{q}_k}{\partial \dot{q}_j} = \delta_{kj}$$

$$\frac{\partial \dot{F}}{\partial \dot{q}_j} = \frac{\partial F}{\partial q_j}$$

ONLY true if F does not depend on $\dot{q}_j$ explicitly !

$$F\left(q_j, t\right)$$

More on Indices and Derivatives:

#2: $$\frac{d}{dt}\left(\frac{\partial F}{\partial q_j}\right) = \frac{\partial}{\partial q_j}\left(\frac{dF}{dt}\right)$$ → Switching is ok here because....

$$\frac{d}{dt}\left(\frac{\partial F}{\partial q_j}\right) = \frac{\partial}{\partial t}\left(\frac{\partial F}{\partial q_j}\right) + \sum_k \frac{\partial}{\partial q_k}\left(\frac{\partial F}{\partial q_j}\right)\dot{q}_k$$

$$= \frac{\partial}{\partial q_j}\left(\frac{\partial F}{\partial t}\right) + \sum_k \frac{\partial}{\partial q_j}\left(\frac{\partial F}{\partial q_k}\right)\dot{q}_k$$

$$= \frac{\partial}{\partial q_j}\left(\frac{\partial F}{\partial t} + \sum_k \frac{\partial F}{\partial q_k}\dot{q}_k\right)$$

$$= \frac{\partial}{\partial q_j}\left(\frac{dF}{dt}\right)$$

Switching is only VALID for partial derivatives in general!

To "factor out" $\partial / \partial q_j$, we also need $\partial \dot{q}_k / \partial q_j = 0$ (later).

$$F\left(q_j, t\right)$$

More on Indices and Derivatives:

#2x: $$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_j}\right) \neq \frac{\partial}{\partial \dot{q}_j}\left(\frac{dF}{dt}\right)$$ → Switching is NOT ok because....

$$\frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_j}\right) = 0$$

F dose not depend on any $\dot{q}_j$ explicitly !

$$\frac{\partial}{\partial \dot{q}_j}\left(\frac{dF}{dt}\right) = \frac{\partial}{\partial \dot{q}_j}\left[\sum_k \frac{\partial F}{\partial q_k}\dot{q}_k + \frac{\partial F}{\partial t}\right]$$

$$= \sum_k \cancel{\frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial F}{\partial q_k}\right)}\dot{q}_k + \left(\frac{\partial F}{\partial q_k}\right)\left(\frac{\partial \dot{q}_k}{\partial \dot{q}_j}\right) + \cancel{\frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial F}{\partial t}\right)}$$

$$= \frac{\partial F}{\partial q_j} \neq \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{q}_j}\right) = 0$$

When in doubt → write them out explicitly !

More on Indices and Derivatives:

#2xx: Q 1.7 → Switching is NOT ok for the following also

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) \neq \frac{\partial}{\partial \dot{q}_j}\left(\frac{dT}{dt}\right)$$

where $T\left(q_j, \dot{q}_j, t\right)$

Note #3:

$$\frac{\partial \dot{q}_j}{\partial q_k} = 0$$

- Typically, we have $\left(q_j, \dot{q}_j, t\right)$ as our standard set of *independent* variables. Some function might depend on all or a subset of them.

- $q_j\left(t\right)$ & $\dot{q}_j\left(t\right)$ are typically considered to be a function of time only unless specifically specified.

Q 1.7:

Starting with the Nelson's form of the E-L Eq,

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} = Q_j \qquad j = 1, \cdots, n$$

→ a scalar equation for each of the generalized coordinates

We first evaluate,

note: $\dfrac{d}{dt} \neq \dfrac{\partial}{\partial t}$

$$\dot{T} = \frac{dT}{dt} = \sum_{i=1}^{n}\left[\frac{\partial T}{\partial q_i}\dot{q}_i + \frac{\partial T}{\partial \dot{q}_i}\ddot{q}_i\right] + \frac{\partial T}{\partial t} \qquad T = T\left(q_i, \dot{q}_i, t\right)$$

or $$\frac{dT}{dt} = \frac{\partial T}{\partial q_i}\dot{q}_i + \frac{\partial T}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial T}{\partial t}$$ (in **Einstein's notations**)

Q 1.7:

Then, we take the partial of $\dot{T}$ with respect to $\dot{q}_j$,

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} = \boxed{\frac{\partial}{\partial \dot{q}_j}\left\{\sum_{i=1}^{n}\left[\frac{\partial T}{\partial q_i}\dot{q}_i\right.\right.} + \left.\left.\frac{\partial T}{\partial \dot{q}_i}\ddot{q}_i\right] + \frac{\partial T}{\partial t}\right\}$$

$$\frac{\partial \dot{q}_i}{\partial \dot{q}_j} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

$$\sum_i \left[\frac{\partial T}{\partial q_i}\boxed{\frac{\partial \dot{q}_i}{\partial \dot{q}_j}} + \frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial T}{\partial q_i}\right)\dot{q}_i\right]$$

$$= \frac{\partial T}{\partial q_j} + \frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial T}{\partial q_i}\right)\dot{q}_i \quad \text{(in Einstein's notations)}$$

Q 1.7:

Similarly, we take partial of the other two terms wrt to $\dot{q}_j$,

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\left\{\sum_{i=1}^{n}\left[\frac{\partial T}{\partial q_i}\dot{q}_i \boxed{+\frac{\partial T}{\partial \dot{q}_i}\ddot{q}_i\right] + \frac{\partial T}{\partial t}}\right\}$$

$$\frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\ddot{q}_i + \frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial T}{\partial t}\right)$$

(in Einstein's notations)

note: $\frac{\partial \ddot{q}_i}{\partial \dot{q}_j} = 0 \quad \forall i$

So, finally, we have,

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} = \frac{\partial T}{\partial q_j} + \frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial T}{\partial q_i}\right)\dot{q}_i + \frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\ddot{q}_i + \frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial T}{\partial t}\right)$$

Q 1.7:

Focusing on the last three terms,

$$\frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial T}{\partial q_i}\right)\dot{q}_i + \frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial T}{\partial \dot{q}_i}\right)\ddot{q}_i + \frac{\partial}{\partial \dot{q}_j}\left(\frac{\partial T}{\partial t}\right)$$

Switch the order of the partial derivatives,

$$\frac{\partial}{\partial q_i}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\dot{q}_i + \frac{\partial}{\partial \dot{q}_i}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\ddot{q}_i + \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}_j}\right)$$

This is in the same form of the full time derivative of $\dfrac{\partial T}{\partial \dot{q}_j}$, i.e.,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = \frac{\partial}{\partial q_i}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\dot{q}_i + \frac{\partial}{\partial \dot{q}_i}\left(\frac{\partial T}{\partial \dot{q}_j}\right)\ddot{q}_i + \frac{\partial}{\partial t}\left(\frac{\partial T}{\partial \dot{q}_j}\right)$$

Q 1.7:

So, collapsing the last three terms into $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)$ we have,

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} = \frac{\partial T}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right)$$

Putting it back into the Nelson's form, we then have,

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} - 2\frac{\partial T}{\partial q_j} = \frac{\partial T}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - 2\frac{\partial T}{\partial q_j} = Q_j$$

This immediately gives the standard form of the E-L Eq,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

Summary of Reminders:

- Pay attention in doing $\dfrac{d}{dt}$ and $\dfrac{\partial}{\partial t}$

- Pay attention to what are the independent variables for a function, e.g.,

$$f = f\left(q_1,\cdots,q_N,\dot{q}_1,\cdots,\dot{q}_N,t\right)$$

→ Obviously, $\left\{q_1,\cdots,q_n,\dot{q}_1,\cdots,\dot{q}_n,t\right\}$ will *implicitly* depend on each other through f but they are NOT *explicitly* depended on each other.

$$\frac{\partial q_1}{\partial q_2} = \frac{\partial \dot{q}_2}{\partial q_1} = \frac{\partial q_1}{\partial \dot{q}_1} = \cdots = 0$$

→ Using indices as shorthand notations:

$$f = f\left(q_j,\dot{q}_j,t\right)$$

with the understanding that $j = 1,\cdots,n$

Summary of Reminders:

→ In taking total derivatives of the function, remember to include all dependences, e.g., with $F = F\left(q_j, \dot{q}_j, t\right)$

$$\frac{dF}{dt} = \sum_{j=1}^{n}\left[\frac{\partial F}{\partial q_j}\frac{dq_j}{dt}\right] + \sum_{j=1}^{n}\left[\frac{\partial F}{\partial \dot{q}_j}\frac{d\dot{q}_j}{dt}\right] + \frac{\partial F}{\partial t}$$

→ A shorthand notation using the **Einstein's notations**, we can use

$$\frac{dF}{dt} = \frac{\partial F}{\partial q_j}\dot{q}_j + \frac{\partial F}{\partial \dot{q}_j}\ddot{q}_j + \frac{\partial F}{\partial t}$$

with the understanding that "repeated index in a given term means sum"

→ In general, you CANNOT switch order in $\frac{d}{dt}\frac{\partial F}{\partial q} \leftrightarrow \frac{\partial}{\partial q}\frac{dF}{dt}$

Summary of Reminders:

- Pay attention to indices, as in the following expression

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\left\{\sum_{i=1}^{n}\left[\frac{\partial T}{\partial q_i}\dot{q}_i + \frac{\partial T}{\partial \dot{q}_i}\ddot{q}_i\right] + \frac{\partial T}{\partial t}\right\}$$

or

$$\frac{\partial \dot{T}}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}\left\{\frac{\partial T}{\partial q_i}\dot{q}_i + \frac{\partial T}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial T}{\partial t}\right\}$$

in Einstein's notation

→ i is the dummy running index for the sum and j is a free index of the expression

Physics Related notes:

HW1.3 (Newton's 2nd & 3rd Laws and logic flow)

- Newton's 2nd Law for two particles

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i^{net}$$

$$m_1 \ddot{\mathbf{r}}_1 = \mathbf{F}_1^e + \mathbf{F}_{21}$$

$$m_2 \ddot{\mathbf{r}}_2 = \mathbf{F}_2^e + \mathbf{F}_{12}$$

- Add the two equations to calculate total momentum for the system:

$$m_1 \ddot{\mathbf{r}}_1 + m_2 \ddot{\mathbf{r}}_2 = \mathbf{F}_1^e + \mathbf{F}_{21} + \mathbf{F}_2^e + \mathbf{F}_{12}$$

$$\Rightarrow \quad M\ddot{\mathbf{R}} = \mathbf{F}_1^e + \mathbf{F}_2^e + \mathbf{F}_{21} + \mathbf{F}_{12} = \mathbf{F}_{tot}^e + \mathbf{F}_{21} + \mathbf{F}_{12}$$

- Now, we want Eq. 1.22, Newton's 2nd Law for a system of particles, to be true,

$$M\ddot{\mathbf{R}} = \mathbf{F}_{tot}^e \quad \Rightarrow \quad \mathbf{F}_{21} + \mathbf{F}_{12} = 0$$

Work-energy theorem: the effect of varying mass

$$W_{12} = \int_1^2 \mathbf{F} \cdot \mathbf{dx} = \int_1^2 m \frac{d\mathbf{v}}{dt} \cdot \mathbf{v} dt = \int_1^2 \frac{m}{2} d\left(v^2\right) = \frac{m}{2}\left(v_2^2 - v_1^2\right)$$ (only for constant *m*)

→ $W_{rocket} \neq \Delta K_{rocket}$ for *m* being a function of time

(check out : R. Newburgh, "Work-energy theorem: the effect of varying mass")

LECTURE REVIEW

D'Alembert's Principle

$$\sum_i \left(\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i \right) \cdot \delta \mathbf{r}_i = 0$$

To solve for EOM using the D'Alembert's Principle …

We need to look into changing variables to a set of *independent* **generalized coordinates** so that we have

$$\sum_j (?)_j \cdot \delta q_j = 0$$

Then, we can claim the coefficients $(?)_j$ in the sum to be independently equal to zero and the **Euler-Lagrange equation** will give an explicit expression for the EOM as:

$$(?)_j = 0$$

E-L Equation for Conservative Forces

Case 1: $\mathbf{F}_i^{(a)}$ derivable from a scalar potential

$$\mathbf{F}_i^{(a)} = -\nabla_i U\left(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N, t\right)$$ (note: U not depend on velocities)

"Generalized Forces"

$$Q_j \equiv \sum_i \mathbf{F}_i^{(a)} \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \Longrightarrow Q_j = -\frac{\partial U}{\partial q_j}$$

E-L Equation for Conservative Forces

We then define the Lagrangian function $L = T - U$ and get the desired

Euler-Lagrange's Equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

Note: there is no unique choice of L which gives a particular set of equations of motion. If $G(q, t)$ is any differentiable function of the generalized coordinate, then

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{dG(q,t)}{dt}$$

will be a different Lagrangian resulting in the same EOM.

E-L Equation for Velocity Dependent Forces

Case 2: U is velocity-dependent, i.e., $U(q_j, \dot{q}_j, t)$

In this case, we redefine the generalize force as,

$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

Then, one can have the same form for the Lagrange's Equation with $L = T - U$,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

E-L Equation for General Forces

Case 3 (General): Applied forces can't be derived from a potential

In general, one can still write down the Lagrange's Equation as,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=Q_j$$

Here,

- L contains the potential from conservative forces as before and
- Q_j represents the forces not arising from a potential

Calculus of Variations: The Problem

Determine the function $y(x)$ such that the following integral is min(max)imized.

$$I = \int_{x_A}^{x_B} F\left\{y(x), y'(x); x\right\} dx$$

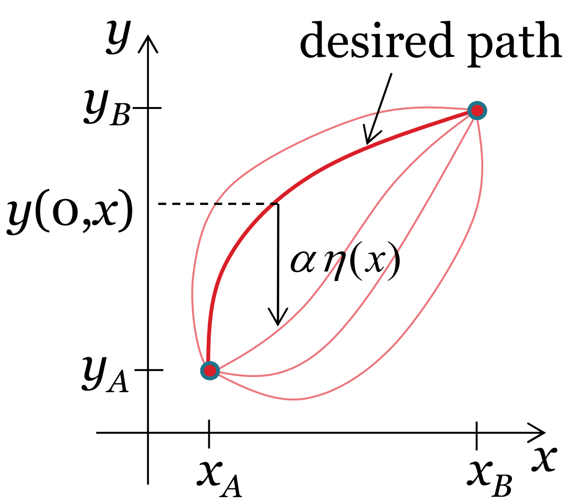
Comments:

1. Notation:
   - x is the *independent* variable *(*$x = t$ in our mech prob*)*
   - $y(x)$ is a function of x and $y'(x) = dy/dx$
   - $F\{\ \}$ is called a **functional** on $y(x)$

2. Intuition: Let say $y(x)$ denote a route in the x-y plane and I is the amount of gas needed for the trip. The problem is to find the route which uses the least gas.

Calculus of Variations

Consider a family of functions $\{y(\alpha, x)\}$ parameterized by α.



$$y(\alpha, x) = y(0, x) + \alpha\eta(x)$$

- α is a parameter
- $\eta(x)$ is a C^2-smooth variation at x
- $\eta(x_A) = \eta(x_A) = 0$

Note: $\left. \begin{array}{l} y(\alpha, x_A) = y_A \\ y(\alpha, x_B) = y_B \end{array} \right\}$ for all α, i.e., all sampled paths has the same end points, x_A and x_B

$y(0, x)$ is the desired path

Calculus of Variations

A necessary, but not sufficient, condition for the min(max)imization of I is:

$$\left.\frac{dI}{d\alpha}\right|_{\alpha=0} = 0 \qquad \text{where} \qquad I = \int_{x_A}^{x_B} F\{y(x), y'(x); x\}dx$$

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$$\boxed{\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0}$$

Euler-Lagrange Equation (1744)
(fundamental lamma of Calc. of Var.)

This is a diff eq whose solution $y(x)$ gives the function we seek to min(max)imize I.